The Input-Output Relationship, or Production Function

- Production is any activity that creates present or future utility
- Another definition for production is a process that transforms inputs into outputs
- A production function is the relationship that describes how inputs like capital and labor are transformed into output
- In a production function that employs only two inputs, capital \((K)\) and labor \((L)\), to produce \(Q\), their relationship may be expressed like:

\[ Q = F(K, L) \text{ Where } F \text{ is a mathematical function} \]

Intermediate Products

- Intermediate products are products which are in the process of being transformed into finished goods

Fixed and Variable inputs

- The production function tells us how output will vary if some or all the inputs are varied
- The long run for a particular production process is defined as the shortest period of time required to alter the amounts of all inputs used in a production process
- The short run of a production process is defined as the longest period of time during which at least one of the inputs used in a production process cannot be varied
- Variable input is an input that can be varied in the short run
- Fixed input is an input that cannot vary in the short run
- The idea that every product will be variable in the long run is vital

Production in the Short Run

- Short-run production functions tend to have a very characteristic shape. They start at the origin, and then move up at an increasing rate, finally the additional input will give a smaller and smaller output, and finally, alter additional input gives no more output, every additional piece of input can even decrease output
- The property that output initially grows at an increasing rate may stem from the benefits of division of tasks and specialization of labor
- The property that output grows at a diminishing rate with the increase in the variable inputs is known as the law of diminishing returns
- The law of diminishing returns states that if other inputs are fixed, the increase in output from an increase in variable input must eventually decline
Total, Marginal, and Average Products

- Short-run production functions are often referred to as total product curves
- A total product curve is a curve showing the amount of output as a function of the amount of variable input
- The marginal product of a variable input is defined as the change in the total product that occurs in response to a unit change in the variable input (all other inputs held fixed)
- The marginal product of labor \( (L) \), denoted \( MP_L \), is defined as:
  \[
  MP_L = \frac{\Delta Q}{\Delta L}
  \]
- The marginal product at any point is simply the slope of the total product curve at any point
- The average product is the total output \( (Q) \) divided by the quantity of the variable input (i.e. \( L \))
  \[
  AP_L = \frac{Q}{L}
  \]
- The average product is the slope of the line joining the origin to the corresponding point on the total product curve

The Relationships among Total, Marginal, and Average Product Curves

- Systematic relationships exist among total, marginal and average products
- Whenever the last unit of an activity exceeds the average level of the activity, the average must be rising. Conversely, when the last nit is smaller than the average, the average must be falling
- When the marginal product curve lies above the average product curve, the average product curve must be rising; vise versa. The two curves intersect at the maximum value of the average product curve
- The result that MP intersects AP at the maximum value of AP can also be shown by nothing that the necessary conditions for a maximum of AP is that its first partial derivative with respect to \( L \) is zero:
  \[
  \frac{\partial}{\partial L} \left( \frac{Q}{L} \right) = \left[ L \frac{\partial Q}{\partial L} \right] \frac{1}{L^2} - Q = 0
  \]

The Practical Significance of the Average-Marginal Distinction

- The distinction between average and marginal product becomes important when one has to allocate resources between two or more activities
- The general rule for allocating an input efficiently in cases of production allocation (where resources are not perfectly divisible, e.g. boats) is to allocate the next unit of the input to the production activity where its marginal product is highest. When the resources are perfectly divisible, then allocate the
resource so that its marginal product is the same in every activity

- an interior solution is one in which each of the production activities is actually employed

Production in the Long Run

- In the long run, all factors of production are variable
- Isoquants are all combinations of variable inputs that yield a given level of output
- The isoquant maps provide a representation of a production process and are very similar do indifference curves
- Isoquants measure the input of one variable on one axis and the input of another one on the other axis

The Marginal rate of Technical Substitution

- The marginal rate of technical substitution (MRTS) is the rate at which one input can be exchanged for another without altering the total level of output
- The MRTS is defined as the absolute value of the slope of the isoquant of two variable inputs. E.g.: 

\[
\left| \frac{\Delta K}{\Delta L} \right|
\]

- For most production functions, the MRTS shows that holding output constant, and having less of one input, we must add the other input to compensate for a one-unit reduction in the first input
- Since the reduction in output of having less of one input (K) is exactly offset by the gain in the output from having more of another output (L):

\[
MP_K \Delta K = MP_L \Delta L
\]

- which can also be written as:

\[
\frac{MP_L}{MP_K} = \frac{\Delta K}{\Delta L}
\]

Returns to Scale

- The technical property of the production function used to describe the relationship between scale and efficiency is called returns to scale
- The concept of returns to scale is an inherently long-run concept
- Increasing returns to scale describes the property of a production process whereby a proportional increase in every input yields a more than proportional increase in output
- Constant returns to scale describes the property of a production process whereby a proportional increase in every input yields an equal proportion increase in output
- Decreasing returns to scale describes the property of a production process whereby a proportional increase in every input yields a less than proportional increase in output
Showing Returns to Scale on the Isoquant Map
- A simple relationship exists between a production function’s return to scale and the spacing of isoquants
- In an isoquant map, increased returns to scale are shown by an equally proportional increase of both inputs, but a much higher increase of quantitative output between different isoquants

The Distinction between diminishing Returns and Decreasing Returns to Scale
- *It is important to bear in mind that decreasing returns to scale have nothing whatsoever to do with the law of diminishing returns*
- Decreasing returns to scale refers to what happens when *all* inputs are varied by a given proportion, while diminishing returns, refers to the case in which one input varies while all others are held fixed

Appendix Chapter 9 – Mathematical Extensions of Production Theory

Application: The Average Marginal Distinction
- People tend to use patterns in many different sorts of things like sports, where they try to use the *average product* to equate their actions (e.g. kicking to the left or right corner of the goal)

Some Examples of Production Functions
The Cobb-Douglas Production Function
- The most widely used production function of all is the Cobb-Douglas which in the two input case takes the form:

\[ Q = mK^\alpha L^\beta \]

- Where \( \alpha \) and \( \beta \) are numbers between zero and one, and \( m \) can be any positive number
- To generate an equation for \( Q \) isoquant, we fix \( Q \) at \( Q_0 \) and then solve for \( K \) in terms of \( L \):

\[ K = \left( \frac{m}{Q_0} \right)^{\frac{1}{\alpha}} (L)^{-\frac{\beta}{\alpha}} \]

- The equations for the marginal products of labor and capital in the Cobb-Douglas are:

\[ MP_K = \frac{\partial Q}{\partial K} = \alpha m K^{\alpha-1} L^\beta \quad \text{And} \quad MP_L = \frac{\partial Q}{\partial L} = \beta m K^\alpha L^{\beta-1} \]

The Leontief, or Fixed-Proportions, Production Function
- The simplest among all production functions that are widely used is the Leontief:

\[ Q = \min(aK, bL) \]

- Its interpretation is simply that \( Q \) is equal to either \( aK \) or \( bL \), whichever is smaller
- the MRS in the Leontief case will be infinite on the vertical arm of the isoquant, zero on the horizontal, and undefined at the cusp in the case of perfect complements
A Mathematical Definition of Returns to Scale

- To increase all inputs in the same proportion means simply to multiply all inputs by the same number $c > 1$
- The definitions of the three cases (increasing, constant, decreasing returns to scale) may be summarized as follows:

  Increasing returns: $F(cK, cL) > cF(K, L)$
  Constant returns: $F(cK, cL) = cF(K, L)$
  Decreasing returns: $F(cK, cL) < cF(K, L)$